

Regularized maximum of strictly plurisubharmonic functions on an almost complex manifold

Alexandre Sukhov*

* Université des Sciences et Technologies de Lille, Laboratoire Paul Painlevé, U.F.R. de Mathématique, 59655 Villeneuve d'Ascq, Cedex, France, sukhov@math.univ-lille1.fr The author is partially supported by Labex CEMPI.

Abstract. We prove that the maximum of two smooth strictly plurisubharmonic functions on an almost complex manifold can be uniformly approximated by smooth strictly plurisubharmonic functions.

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1 Introduction

In this note the following result is proved:

Proposition 1.1 *Let u_1 and u_2 be smooth strictly J -plurisubharmonic functions on an almost complex manifold (M, J) . Then for every $\varepsilon > 0$ and every relatively compact domain $\Omega \subset M$ there exists a function \tilde{u} smooth and strictly J -plurisubharmonic in Ω such that*

$$\max\{u_1, u_2\} \leq \tilde{u} \leq \max\{u_1, u_2\} + \varepsilon \quad (1)$$

on Ω .

This property of plurisubharmonic functions is standard in the case where an almost complex structure J is integrable. In their recent lecture notes Cieliebak and Eliashberg [2] raised this question in the almost complex case. Proposition 1.1 gives an affirmative answer. It is a corollary of a more precise result (Theorem 3.2). The elementary proof is proceeded by the known construction of the regularized maximum function (see the monograph of Demailly [3]) and requires only minor modifications with respect to the integrable case.

2 Plurisubharmonic functions on almost complex manifolds: the background

For convenience of reader's I recall related notions.

2.1. Almost complex structures. Consider a smooth (everywhere this means of class C^∞) manifold M of dimension $2n$. An almost complex structure J is a smooth map assigning to each point $p \in M$ a linear isomorphism $J(p) : T_p M \rightarrow T_p M$ of the tangent space $T_p M$ such that $J(p)^2 = -I$; here $I : T_p M \rightarrow T_p M$ denotes the identity map. A couple (M, J) is called an almost complex manifold of complex dimension n . The present paper concerns with almost complex manifolds and structures of class C^∞ though the results still hold under a lower regularity.

A C^1 -map $f : M' \rightarrow M$ between two almost complex manifolds (M', J') and (M, J) , is called (J', J) -complex or (J', J) -holomorphic if it satisfies the *Cauchy-Riemann equations* $df \circ J' = J \circ df$. By the elliptic regularity, such a map is necessarily of class C^∞ . If the source manifold M' is a Riemann surface, the holomorphic maps are called J -complex (or J -holomorphic) curves. Denote by \mathbb{D} the unit disc in \mathbb{C} and by J_{st} the standard complex structure of \mathbb{C}^n ; the value of n will be clear from the context. In the case where $M' = \mathbb{D}$ and $J' = J_{st}$, a J -holomorphic map f is called a *J -complex disc*.

2.2. Local coordinates. Let \mathbb{B}_n denotes the Euclidean unit ball of \mathbb{C}^n . For every point p in an almost complex manifold M of complex dimension n , there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \rightarrow \mathbb{B}_n$ with $z(p) = 0$, such that the direct image $z_*(J) := dz \circ J \circ dz^{-1}$ satisfies $z_*(J)(0) = J_{st}$. Choose an integer $k \geq 1$ and $\lambda_0 > 0$. Composing z with isotropic dilations in \mathbb{C}^n one can additionally achieve the condition $\|z_*(J) - J_{st}\|_{C^k(\overline{\mathbb{B}_n})} \leq \lambda_0$.

In these local coordinates J is represented by a \mathbb{R} -linear operator $J(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z \in \mathbb{C}^n$ such that $J(z)^2 = -I$. We use the notation $\zeta = \xi + i\eta \in \mathbb{D}$ for the standard complex coordinate in \mathbb{C} . Then the Cauchy-Riemann equations for a J -complex disc $f : \mathbb{D} \rightarrow \mathbb{B}_n$ have the form $\partial_{\eta} f = J(f) \partial_{\xi} f$. Similarly to [1], the structure J defines a unique smooth complex $n \times n$ matrix function $A_J = A_J(z)$ allowing to write the Cauchy-Riemann equations as an elliptic quasilinear deformation of the usual $\bar{\partial}$ -equation:

$$\partial_{\bar{\zeta}} f + A_J(f) \partial_{\bar{\zeta}} \bar{f} = 0 \quad (2)$$

A_J is called the *complex matrix* of J in the coordinates z . If $z' : U \rightarrow \mathbb{C}^n$ is another coordinate chart and A'_J is the complex matrix of J in the coordinates z' , then $A'_J = ((\partial_z z') A_J - \partial_{\bar{z}} z') (\partial_{\bar{z}} \bar{z}' - (\partial_z \bar{z}') A_J)^{-1}$ (see, for example, [4]). The condition $J(0) = J_{st}$ means that

$$A_J(0) = 0 \quad (3)$$

As it was mentioned, given $k \geq 0$ the norm $\|A_J\|_{C^k(\mathbb{B}_n)}$ can be made arbitrarily small by an isotropic dilation of coordinates.

2.3. Plurisubharmonic functions. Let u be a real C^2 function on an open subset Ω of an almost complex manifold (M, J) . Denote by $J^* du$ the differential form acting on a vector field X by $J^* du(X) := du(JX)$. Given point $p \in M$ and a tangent vector $V \in T_p(M)$ consider a smooth vector field X in a neighborhood of p satisfying $X(p) = V$. The value of the *complex Hessian* (or the Levi form) of u with respect to J at p and V is defined by $H(u)(p, V) := -(dJ^* du)_p(X, JX)$.

This definition is independent of the choice of a vector field X . For instance, if $J = J_{st}$ in \mathbb{C} , then $-dJ^*du = \Delta u d\xi \wedge d\eta$; here Δ denotes the Laplacian. In particular, $H_{J_{st}}(u)(0, \frac{\partial}{\partial \xi}) = \Delta u(0)$.

Recall some basic properties of the complex Hessian (see for instance, [4]):

Lemma 2.1 *Consider a real function u of class C^2 in a neighborhood of a point $p \in M$.*

- (i) *Let $F : (M', J') \rightarrow (M, J)$ be a (J', J) -holomorphic map, $F(p') = p$. For each vector $V' \in T_{p'}(M')$ we have $H_{J'}(u \circ F)(p', V') = H_J(u)(p, dF(p)(V'))$.*
- (ii) *If $f : \mathbb{D} \rightarrow M$ is a J -complex disc satisfying $f(0) = p$, and $df(0)(\frac{\partial}{\partial \xi}) = V \in T_p(M)$, then $H_J(u)(p, V) = \Delta(u \circ f)(0)$.*

Property (i) expresses the holomorphic invariance of the complex Hessian. Property (ii) is often useful in order to compute the complex Hessian on a given tangent vector V .

Let Ω be a domain M . An upper semicontinuous function $u : \Omega \rightarrow [-\infty, +\infty[$ on (M, J) is *J -plurisubharmonic* if for every J -complex disc $f : \mathbb{D} \rightarrow \Omega$ the composition $u \circ f$ is a subharmonic function on \mathbb{D} . By Proposition 2.1, a C^2 function u is plurisubharmonic on Ω if and only if it has a positive semi-definite complex Hessian on Ω i.e. $H_J(u)(p, V) \geq 0$ for any $p \in \Omega$ and $V \in T_p(M)$. The equivalence of these two definitions still holds in the general case if the complex Hessian is understood in the sense of currents. This was established by Pali [6] for continuous functions and by Harvey-Lawson [5] for upper semicontinuous functions.

A real C^2 function $u : \Omega \rightarrow \mathbb{R}$ is called *strictly J -plurisubharmonic* on Ω , if $H_J(u)(p, V) > 0$ for each $p \in \Omega$ and $V \in T_p(M) \setminus \{0\}$. Obviously, these notions are local: an upper semicontinuous (resp. of class C^2) function on Ω is J -plurisubharmonic (resp. strictly) on Ω if and only if it is J -plurisubharmonic (resp. strictly) in some open neighborhood of each point of Ω .

As above, choosing local coordinates near p we may identify a neighborhood of p with a neighborhood of the origin, assuming that J -complex discs are solutions of the equations (2) and the condition (3) holds. The following technical result shows that after an additional change of local coordinates one can achieve a further normalization of the complex matrix A_J of J .

Lemma 2.2 *There exists local coordinate diffeomorphism fixing the origin whose components are polynomials of degree at most 2 and with the linear part equal to the identity map, such that in the new coordinates z the following holds:*

- (i) *the matrix function A_J from the equation (2) still satisfies (3) and additionally satisfies the condition*

$$\partial_z A_J(0) = 0 \tag{4}$$

- (ii) *Every real function $u = u(z)$ of class C^2 satisfies*

$$H_J(u)(0; V) = H_{J_{st}}(u)(0; V) = \sum_{k,j=1}^n \frac{\partial^2 u}{\partial z_k \partial \bar{z}_j}(0) V_k \bar{V}_j \tag{5}$$

for each vector $V = (V_1, \dots, V_n) \in \mathbb{C}^n$.

I learned this result from unpublished lecture notes of Chirka although it is possible that it was known before. An elementary proof can be found, for instance, in [4].

Local coordinates given by Lemma 2.2 near a point $p \in M$ are called *adapted coordinates* at p . The existence of adapted local coordinates provides a convenient way to evaluate the complex Hessian of a C^2 function at a given point of an almost complex manifold.

Conclude this section by two remarks.

1. The condition (3) alone (without (4)) in general does not guarantee the property (5). Consider, for example, the harmonic function $u(\zeta) = \operatorname{Re} \zeta$ in a neighborhood of the origin in \mathbb{C} . After the local change of coordinates $\zeta = F(\tilde{\zeta}) = \tilde{\zeta} + |\tilde{\zeta}|^2$ we obtain the function $\tilde{u}(\tilde{\zeta}) = u(F(\tilde{\zeta})) = \operatorname{Re} \tilde{\zeta} + |\tilde{\zeta}|^2$. Since $dF(0) = I$, the complex structure $J = (F)^*(J_{st}) := (F^{-1})_*(J_{st})$ satisfies the condition (3). However \tilde{u} is not strictly J -plurisubharmonic since $H_J(\tilde{u})$ vanishes identically in a neighborhood of the origin.

2. Denote by $\|\bullet\|$ the Euclidean norm on \mathbb{C}^n . In the adapted coordinates the function $u(z) = \|z\|^2$ is strictly J -plurisubharmonic and satisfies (5). If local coordinates are not adapted and only the condition (3) holds, then in general the equality (5) can fail. However, even in this case u is still strictly J -plurisubharmonic near the origin if the norm $\|A\|_{C^1}$ is small enough.

3 Regularized max-function

In this section Proposition 1.1 is proved. I literally follow the presentation given by Demailly [3].

Denote by $d\mu^k$ the standard Lebesgue measure on \mathbb{R}^k . Fix a C^∞ function $\omega : \mathbb{R} \rightarrow \mathbb{R}^+$ such that the support of ω is contained in $[0, 1]$ and $\int_{\mathbb{R}} \omega d\mu^1 = 1$.

Given $\theta = (\theta_1, \dots, \theta_k) \in (\mathbb{R}^+)^k$ consider the *regularized max-function*:

$$M_\theta(t_1, \dots, t_k) = \int_{\mathbb{R}^k} \max\{t_1 + s_1, \dots, t_k + s_k\} \prod_{j=1}^k \theta_j^{-1} \omega(s_j / \theta_j) d\mu^k(s) \quad (6)$$

defined for $t = (t_1, \dots, t_k) \in \mathbb{R}^k$. Its utility arises from the following

Lemma 3.1 *For every $\theta \in (\mathbb{R}^+)^k$ we have:*

- (i) *The function $t = (t_1, \dots, t_k) \mapsto M_\theta(t_1, \dots, t_k)$ is smooth on \mathbb{R}^k .*
- (ii) *$\max\{t_1, \dots, t_k\} \leq M_\theta(t_1, \dots, t_k) \leq \max\{t_1 + \theta_1, \dots, t_k + \theta_k\}$ for every $t \in \mathbb{R}^k$.*
- (iii) *$M_\theta(t_1 + a, \dots, t_k + a) = M_\theta(t_1, \dots, t_k) + a$ for every $a \in \mathbb{R}$ and every $t \in \mathbb{R}^k$.*
- (iv) *Let u_1, \dots, u_k be smooth J -plurisubharmonic functions on M . Then $M_\theta(u_1, \dots, u_k)$ is a smooth J -plurisubharmonic function on M .*

Proof. (i) Performing the change of variables $t_j + s_j = s'_j$ we obtain

$$M_\theta(t_1, \dots, t_k) = \int_{\mathbb{R}^k} \max\{s'_1, \dots, s'_k\} \prod_{j=1}^k \theta_j^{-1} \omega((s'_j - t_j) / \theta_j) d\mu^k(s')$$

Differentiation of this integral with respect to the parameters t_j shows that M_θ is smooth. (ii) and (iii) follow from the assumptions on the function ω . At (iv), consider a J -complex disc $f : \mathbb{D} \rightarrow M$. For all s_j the composition

$$\mathbb{D} \ni \zeta \mapsto \max\{u_1(f(\zeta)) + s_1, \dots, u_k(f(\zeta)) + s_k\}$$

is a subharmonic function on \mathbb{D} . Therefore the function

$$\mathbb{D} \ni \zeta \mapsto M_\theta(u_1(\zeta), \dots, u_k(\zeta))$$

is subharmonic in \mathbb{D} by Theorem 2.4.8 in [7]. Hence the function $M_\theta(u_1, \dots, u_k)$ is a smooth J -plurisubharmonic function on Ω . ■

Choose a continuous hermitian metric h on (M, J) and denote by $h_p(X)$ the value of h at point $p \in M$ on a vector $V \in T_p M$. Proposition 1.1 is a consequence of the following more precise result.

Theorem 3.2 *Let α be a smooth real function on M . Suppose that u_1, \dots, u_k are smooth functions on Ω satisfying $H_J(u_j)(p, V) \geq \alpha(p)h_p(V)$ for every point $p \in \Omega$ and each vector $V \in T_p M$. Then for every $\theta = (\theta_1, \dots, \theta_k) \in (\mathbb{R}^+)^k$ the function $\tilde{u} = M_\theta(u_1, \dots, u_k)$ is smooth and satisfies $H_J(\tilde{u})(p, V) \geq \alpha(p)h_p(V)$ on Ω .*

Proof. Fix a point p and $\delta > 0$. Choose adapted coordinates at the point p . In particular, in these coordinates p corresponds to the origin. Let us simply write J instead of $(z)_*(J)$. Since the coordinates are adapted, the function $z \mapsto h_0(z)$ is a positive definite J_{st} -hermitian form and $H_J(h_0)(0, V) = H_{J_{st}}(h_0)(0, V) = h_0(V)$. Consider the smooth functions $v_j(z) = u_j(z) - \alpha(0)h_0(z) + \delta \|z\|^2$. They satisfy $H_J(v_j)(0, V) = H_{J_{st}}(v_j)(0, V) \geq \delta \|V\|^2$. Hence they are J -plurisubharmonic in a neighborhood of the origin (in general, depending on δ). By (iv) of Lemma 3.1 the smooth function $\tilde{v} := M_\theta(v_1, \dots, v_k)$ is J -plurisubharmonic near the origin as well and by (iii) Lemma 3.1 one has $\tilde{v} = \tilde{u} - \alpha(0)h_0(z) + \delta \|z\|^2$. Since $\delta > 0$ is arbitrary, we obtain that $H_J(\tilde{u})(0, V) \geq \alpha(0)h_0(V)$. ■

In order to prove Proposition 1.1 it suffices now for a given $\varepsilon > 0$ to choose $\theta_j = \varepsilon$, $j = 1, 2$ and to apply Theorem 3.2 with α equal to the constant function 1. The property (ii) of Lemma 3.1 implies the estimate (1). The proof is completed.

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